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THE STRUCTURE OF COMPACT RICCI-FLAT RIEMANNIAN MANIFOLDS

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0. Introduction and preliminaries

An interesting problem in riemannian geometry is to determine the structure of complete riemannian manifolds with Ricci tensor zero (Ricci-flat). In particular one asks whether such manifolds are flat. Here we show that any compact connected Ricci-flat n-manifold M^n has the expression

$$M^n = \Psi \backslash T^k \times M^{n-k} ,$$

where k is the first Betti number $b_1(M^n)$, T^k is a flat riemannian k-torus, M^{n-k} is a compact connected Ricci-flat $(n - k)$-manifold, and Ψ is a finite group of fixed point free isometries of $T^k \times M^{n-k}$ of a certain sort (Theorem 4.1). This extends Calabi's result on the structure of compact euclidean space forms ([7]; see [20, p. 125]) from flat manifolds to Ricci-flat manifolds. We use it to essentially reduce the problem of the construction of all compact Ricci-flat riemannian n-manifolds to the construction in dimensions $< n$ and in dimension n to the case of manifolds with $b_1 = 0$ (see § 4). We also use it to prove (Corollary 4.3) that any compact connected Ricci-flat manifold M has a finite normal riemannian covering $T \times N \to M$ where T is a flat riemannian torus, $\dim T \geq b_1(M)$, and N is a compact connected simply connected Ricci-flat riemannian manifold. This extends one of the Bieberbach theorems [4], [20, Theorem 3.3.1] from flat manifolds to Ricci-flat manifolds, and reduces the question of whether compact Ricci-flat manifolds are flat to the simply connected case. J. Cheeger and D. Gromoll have pointed out to us that this extension also follows from their proof of [8, Theorem 6]. Our direct proof however uses considerably less machinery than their deeper considerations of manifolds of nonnegative curvature.

As a consequence of these results, we can give a variety of sufficient topological conditions for Ricci-flat riemannian n-manifolds M to be flat. For example, if the homotopy groups $\pi_k(M) = 0$ for $k > 1$, or the universal covering of M is acyclic (Theorem 4.6), or M has a finite topological covering by a

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space with $b_1 \geq n - 3$, then M is flat (Corollaries 4.3 and 4.4). In particular, a given compact manifold which admits a flat riemannian structure satisfies these conditions (see also Corollary 2.5). Thus a given compact manifold cannot have both flat riemannian structures and nonflat Ricci-flat riemannian structures. This remark is useful in studying some subsets of the space of riemannian metrics on a given compact manifold; see [9] and [10].

In § 2 we give some results for riemannian manifolds with positive semi-definite Ricci tensor. We apply these results in § 3 to show that if a compact connected riemannian manifold M with positive semi-definite Ricci curvature is homotopy-equivalent to a generalized nilmanifold, then M is flat (Theorem 3.1). In particular, if M is homotopy-equivalent to a euclidean space form, then M is flat. This theorem sharpens a result of Wolf [19, Theorem 6.4] on generalized nilmanifolds.

After this paper was written, E. Calabi informed us that he had known that the Calabi construction was valid for Ricci-flat manifolds. He refered us to his paper [6] where the kaehler case of our Theorem 4.1 is worked out in the course of the argument of Theorem 1. There, T^k is the Albanese variety of M^n, Calabi's Jacobi map $J: M^n \to T^k$ is both a holomorphic bundle and a riemannian submersion, and the J-fibres correspond to our M^{n-k}.

1. Preliminaries

By "riemannian manifold" we mean a C^∞ hausdorff differentiable manifold without boundary, together with a C^∞ riemannian (positive definite) metric. If M is a riemannian manifold and $p \geq 0$ an integer, then $b_p(M)$ denotes the p-th Betti number for singular cohomology; it is the real dimension of the de Rham group

$$\{p\text{-forms } \omega : d\omega = 0\} / \{d\eta : \eta \text{ is a } (p-1)\text{-form}\} .$$

If M is compact, then the Hodge theorem says that the de Rham group is isomorphic to the space $\mathscr{H}_p = \{p\text{-forms } \omega : \Delta\omega = 0\}$ where $\Delta = d\delta + \delta d$ is the Laplace-de Rham operator. If the riemannian manifold M is not orientable, then $\delta\omega$ is defined by its local coordinate expression: $(\delta\omega)_{i_1\cdots i_{p-1}} = -\nabla^k \omega_{ki_1\cdots i_{p-1}}$ (Einstein summation convention) for $p > 0$, and δ annihilates functions. If M is compact, then the Hodge theorem for M comes down from the two-sheeted orientable riemannian covering manifold $\pi: \tilde{M} \to M$ as follows. Express $M = \Gamma \backslash \tilde{M}$, where $\Gamma = \{1, \gamma\}$ and γ is a fixed point free involutive isometry of $\tilde{M}$. Let ω be a p-form on M with $d\omega = 0$. Express $\pi^*\omega = h(\pi^*\omega) + d\lambda$ where $h(\pi^*\omega)$ is a harmonic p-form on $\tilde{M}$, and λ is a $(p-1)$-form on $\tilde{M}$. Evidently $\gamma^*(\pi^*\omega) = \pi^*\omega$, and also $\gamma^* \cdot h = h \cdot \gamma^*$ because γ is an isometry. Thus $\pi^*\omega = h(\pi^*\omega) + d\pi^*\eta$ where η is the $(p-1)$-form on M defined by $\pi^*\eta = \frac{1}{2}(1 + \gamma^*)\lambda$. Now $\omega = h(\omega) + d\eta$ where $h(\omega)$ is defined to be the p-form on M with π^*-lift

$h(\pi^*\omega)$. Since $h(\pi^*\omega)$ is harmonic on $\tilde{M}$ and the covering is locally isometric, $h(\omega)$ is a harmonic p-form on M. Uniqueness of $h(\pi^*\omega)$ in $\pi^*\omega = h(\pi^*\omega) + d\pi^*\eta$ implies uniqueness of $h(\omega)$ in $\omega = h\omega + d\eta$. Thus $\omega \mapsto h(\omega)$ defines an isomorphism of the p-th de Rham group of M onto the space of harmonic p-forms.

For a development of Hodge theory which does not use orientability, see Nelson [13, § 7].

The Ricci tensor of M is denoted r. Let X be a nonzero tangent vector at a point $x \in M$. The *Ricci curvature* of X *at* x is defined to be $r(X, x)/\|X\|^2$. In local coordinates with the sign convention $R_{ij} = R^m{}_{ijm}$, the Ricci curvature is $R_{ij}X^iX^j/g_{ij}X^iX^j$. The *mean curvature*, i.e., the average of the sectional curvatures for plane sections of M_x which contain X, is $(n-1)^{-1}r(X, X)/\|X\|^2$.

We say that a vector field X is *parallel* if $\nabla X = 0$. This means that if p, $q \in M$ and σ is a curve from p to q, then parallel translation along σ carries X_p to X_q. $\mathscr{X}_{\|}$ denotes the set of all parallel vector fields on M.

The Laplace-de Rham operator acts on vector fields through their correspondence with 1-forms, and we let $\mathscr{H} = \{X : \Delta X = 0\}$, the harmonic vector fields on M. Also, we let $I(M)$ be the isometry group of M, $I(M)^0$ its connected component of the identity, and $\mathscr{I}(M)$ the Lie algebra of Killing vector fields on M. We make extensive use of the following results of Bochner ([5]; see [22, pp. 37 and 39] and [21]):

Theorem 1.1 (*Bochner*). *Let M be a compact riemannian manifold. If X is a harmonic vector field on M with $r(X, X) \geq 0$, then X is parallel and $r(X, X) = 0$. If X is a Killing vector field on M with $r(X, X) \leq 0$, then X is parallel and $r(X, X) = 0$.*

We will refer to these results as "the Bochner lemma". Note that connectedness and orientability are dropped from the usual formulation. If M is nonorientable, then the volume element $d\mu$ formed from the riemannian structure is a measure but not an n-form. However, for M compact, Green's theorem $\int_M \delta X d\mu = 0$ still holds. This is sufficient for the Bochner lemma to apply to nonorientable M.

2. Nonnegative mean curvature

In this section we study compact riemannian manifolds M with every mean curvature ≥ 0, i.e., whose Ricci tensor r is positive semi-definite. We are able to extract some consequences for the Betti numbers of such a manifold; for example, we give the lower bound $b_p(M) \geq \binom{k}{p}$, $k = b_1(M)$ (Theorem 2.3). We then use an idea of Berger to give a sufficient topological condition for such a manifold to be flat (Theorem 2.4). In particular, this condition is satisfied by a compact manifold which admits a flat riemannian structure.

In § 4 we shall give weaker sufficient topological conditions for M to be flat under the stronger geometrical conditions $\boldsymbol{r} = 0$.

Theorem 2.1. *Let M be a compact connected riemannian manifold, and $b_1(M)$ its first Betti number.*

If M has every mean curvature ≥ 0, then $\mathscr{X}_{\|} = \mathscr{H}$, and $\mathscr{X}_{\|}$ is a central ideal in the Lie algebra $\mathscr{I}(M)$ of all Killing vector fields on M. Further $\mathscr{X}_{\|}$ defines a $b_1(M)$-dimensional foliation of M by flat totally geodesic submanifolds.

If M has every mean curvature ≤ 0, then $\mathscr{X}_{\|} = \mathscr{I}(M) \subset \mathscr{H}$, and the identity component $I(M)^0$ of the isometry group is a torus group T of dimension $\leq b_1(M)$. T acts effectively and smoothly on M, and the orbits of the action foliate M by flat totally geodesic tori of the same dimension as T.

Proof. Clearly $\mathscr{X}_{\|} \subset \mathscr{H}$ and $\mathscr{X}_{\|} \subset \mathscr{I}(M)$ for any riemannian manifold.

Suppose that M has mean curvature ≥ 0. Then by the Bochner lemma, $\mathscr{H} \subset \mathscr{X}_{\|}$, so that $\mathscr{X}_{\|} = \mathscr{H} \subset \mathscr{I}(M)$. Also, by Hodge's theorem, $\dim \mathscr{X}_{\|} = \dim \mathscr{H} = b_1(M)$.

If X is parallel and Z is a Killling vector field, then

$$[X, Z] = \nabla_X Z = -\nabla\langle X, Z\rangle = 0 ,$$

since the contraction of a harmonic vector field and a Killing vector field is a constant [5], [22, p. 44]. Thus $\mathscr{X}_{\|}$ is a central ideal in $\mathscr{I}(M)$. In particular, $[\mathscr{X}_{\|}, \mathscr{X}_{\|}] = 0$ so that $\mathscr{X}_{\|}$ defines an involutive distribution of dimension $b_1(M)$. Thus M is foliated by flat $b_1(M)$-dimensional submanifolds. These submanifolds are totally geodesic in M because integral curves of parallel vector fields are geodesics.

If every mean curvature ≤ 0, then by the Bochner lemma, every Killing vector field is parallel. Thus $\mathscr{I}(M) = \mathscr{X}_{\|} \subset \mathscr{H}$. Since $\mathscr{X}_{\|}$ is an abelian Lie algebra, $I(M)^0$ is a torus group T of dimension $\leq b_1(M) = \dim \mathscr{H}$. q.e.d.

In case every mean curvature ≥ 0, the Lie subalgebra $\mathscr{X}_{\|}$ of the center of $\mathscr{I}(M)$ generates an abelian analytic subgroup of $I(M)^0$ whose closure is a central torus subgroup of dimension $\geq \dim \mathscr{X}_{\|} = b_1(M)$. Thus the identity component of the center of $I(M)^0$ is a torus of dimension $\geq b_1(M)$. Also in this case, $\mathscr{X}_{\|}$ defines a smooth effective nonsingular action of the additive group $\boldsymbol{R}^k$ ($k = b_1(M)$) on M, given by

$$\boldsymbol{R}^k \times M \to M , \qquad ((t_1, \cdots, t_k), m) \mapsto F^1_{t_1} \circ \cdots \circ F^k_{t_k}(m) .$$

Here the $\{F^i_{t_i}\}_{1 \leq i \leq k}$ are the respective flows of k linearly independent parallel vector fields. The orbits of this action are the leaves of the theorem.

We also remark that if $b_1(M) \geq 1$ and M has every mean curvature ≥ 0, then M has a parallel and hence nonvanishing vector field. Hence the Euler-Poincaré characteristic $\chi_M = 0$.

In the case where M is Ricci-flat, we know $I(M)^0$ exactly.

Corollary 2.2. *Let M be a compact connected riemannian manifold with first Betti number $b_1(M)$. If M has every mean curvature $= 0$, then $\mathscr{X}_{\|} = \mathscr{H} = \mathscr{I}(M)^0$ and $I(M)^0$ is a b_1-dimensional torus group.*

Proof. Since $r \geq 0$ and $r \leq 0$, by the theorem $\mathscr{X}_{\|} = \mathscr{H} \subset \mathscr{I}(M)$ and $\mathscr{X}_{\|} = \mathscr{I}(M) \subset \mathscr{H}$. q.e.d.

The corollary generalizes the same result for compact flat manifolds. In the flat case, however, $I(M)$ can be explicitly described by the method of [19, proof of Theorem 1].

If M has every mean curvature ≥ 0, then we can extract some consequences concerning the Betti numbers of M.

Theorem 2.3. *Let M be a compact connected n-dimensional manifold with every mean curvature ≥ 0, and $k = b_1(M)$ its first Betti number. Then*

$$b_p(M) \geq \binom{k}{p} \qquad \text{for } p \leq k .$$

Also $b_1(M) \leq n$, and $b_1(M) = n$ if and only if M is a flat riemannian n-torus. If $b_1(M) = n - 1$, then M is flat but not orientable.

Proof. Let $\{X_i\}_{1 \leq i \leq k}$ be k linearly independent parallel vector fields, and let $\{\theta^i\}_{1 \leq i \leq k}$ be the dual 1-forms, $\theta^i(Y) = \langle Y, X_i \rangle$. Now the $\{\theta^i\}$ are parallel and thus harmonic, as are the

$$\theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} , \quad I = (i_1, \cdots, i_p) , \qquad 1 \leq i_1 < \cdots < i_p \leq k .$$

Since these $\{\theta^I\}$ are $\binom{k}{p}$ linearly independent harmonic p-forms, by Hodge's theorem we have $b_p(M) \geq \binom{k}{p}$ for $p \leq k$.

Since $\mathscr{H} = \mathscr{X}_{\|}$, $\dim \mathscr{H} = b_1(M) \leq n$. If $b_1(M) = n$, then M has a parallel frame. The curvature tensor vanishes in this frame, so M is flat. Since $I(M)^0$ is an n-dimensional torus, so is M.

If $b_1(M) = n - 1$, then we have $n - 1$ linearly independent parallel vector fields $\{X_i\}_{1 \leq i \leq n-1}$ on M. If M were orientable, then this could be extended to a frame $\{X_1, \cdots, X_{n-1}, Z\}$, where Z is orthogonal to the $n - 1$ parallel vector fields $\{X_i\}_{1 \leq i \leq n-1}$ and normalized to unity, i.e., $\langle Z, Z \rangle = 1$. Then for any vector field Y and X_i parallel, $\nabla_Y \langle Z, Z \rangle = 2\langle Z, \nabla_Y Z \rangle = 0$, and $\nabla_Y \langle X_i, Z \rangle = \langle X_i, \nabla_Y Z \rangle = 0$. Since $\{X_i, Z\}_{1 \leq i \leq n-1}$ is a frame at each point, $\nabla_Y Z = 0$ for all Y so Z is parallel. Hence M has n linearly independent parallel vector fields so that $b_1(M) = n$, a contradiction. Now M is not orientable, and its 2-sheeted orientable cover is a flat torus.

Remarks. 1. If $b_1(M) = [\frac{1}{2}n]$, then from Poincaré duality the theorem gives a lower bound for all the Betti numbers of M.

2. The example of the n-torus with $b_p = \binom{n}{p}$ shows that our bounds are the best possible (in terms of the first Betti number alone).

3. The conditions on the Betti numbers are necessary topological conditions for a manifold to admit a riemannian structure with every mean curvature ≥ 0 and in particular to admit a flat riemannian structure; cf. also Cheeger-Gromoll [8].

Let $\chi_M(t) = \sum_{i=0}^n b_i(M)t^i$ be the Euler-Poincaré polynomial of M. Lichnerowicz [11] has shown that if every mean curvature ≥ 0, then $\chi_M(t)$ is divisible by $(t + 1)^{b_1(M)}$.

4. It is interesting that for M orientable with every mean curvature ≥ 0 there is a gap in the possible values of the first Betti number. The example of the Klein bottle with $b_1 = 1$ shows that orientability is necessary, and the disjoint union $S^1 \times S^2 \cup S^1 \times S^2$ with $b_1 = 2$ shows that connectedness is necessary.

5. Let span M be the maximal number of vector fields on M which are linearly independent at each point, and let rank M be the maximal number of commuting vector fields which are linearly independent at each point. From Theorem 2.1, if M has every mean curvature ≥ 0, then M has $b_1(M)$ parallel and hence commuting vector fields. Thus span $M \geq$ rank $M \geq b_1(M)$. For M orientable, span $M \geq n - 1 \Rightarrow$ span $M = n$ which is analogous to dim $\mathscr{X}_\parallel \geq n - 1 \Rightarrow$ dim $\mathscr{X}_\parallel = n$. Since an orientable nontrivial 2-torus bundle over a circle is a 3-manifold of rank 2 (a result of H. Rosenberg, R. Roussarie, and D. Weil [14]) rank M does not have this property. Thus an orientable n-manifold can have rank $n - 1$, and then given $n - 1$ commuting vector fields linearly independent at each point there is no riemannian metric in which these vector fields can be made parallel.

Berger [2, § 8] and Berger-Ebin [3, § 8] show that a Ricci-flat variation of a flat riemannian metric remains flat. We generalize this as follows.

Theorem 2.4. *Let M be a compact connected n-dimensional manifold. Suppose that M admits a finite topological covering $\pi : \tilde{M} \to M$ with $b_1(\tilde{M}) = n$. If g is a riemannian structure on M with every mean curvature ≥ 0, then $(\tilde{M}, \pi^*g)$ is a flat riemannian torus and g is a flat riemannian metric on M.*

Proof. Endow $\tilde{M}$ with the differentiable manifold structure for which the covering is differentiable. Let $\tilde{g} = \pi^*g$ be the pull-back of g. Then (M, g) and $(\tilde{M}, \tilde{g})$ are locally isometric, so $\tilde{g}$ has every mean curvature ≥ 0. Since $b_1(\tilde{M}) = n$, $(\tilde{M}, \tilde{g})$ is a flat n-torus by Theorem 2.3, so g is flat.

Corollary 2.5. *If a compact manifold M admits a flat riemannian structure, then every riemannian structure with mean curvature ≥ 0 on M is flat.*

Proof. One of the Bieberbach theorems [4]; [20, Theorem 3.3.1] says that each connected component of M is covered by a torus. q.e.d.

In particular, a compact manifold cannot have both flat riemannian metrics and nonflat Ricci-flat riemannian metrics.

3. Application to generalized nilmanifolds

Let G be a connected Lie group. Then its automorphism group Aut (G) is a real linear algebraic group. The "affine group" $A(G)$ is the semidirect product $G \cdot \mathrm{Aut}\,(G)$, acting on G by $(g, \alpha): x \mapsto g \cdot \alpha(x)$. Aut (G) has maximal compact subgroups, and any two are conjugate. Choose a maximal compact subgroup $K \subset \mathrm{Aut}\,(G)$. The "euclidean group" is the closed subgroup $E(G) = G \cdot K$ in $A(G)$.

If G is the n-dimensional real vector group $\boldsymbol{R}^n$, then Aut $(G) = GL(n, \boldsymbol{R})$, the general linear group, and its maximal compact subgroup is just the orthogonal group $\mathrm{O}(n)$. Then $A(G)$ is the usual affine group, $A(n) = \boldsymbol{R}^n \cdot GL(n, \boldsymbol{R})$, and $E(G)$ is the usual euclidean group $E(n) = \boldsymbol{R}^n \cdot \mathrm{O}(n)$.

A differentiable manifold M is called a *generalized nilmanifold* if it is diffeomorphic to a quotient $\Gamma \backslash N$, where N is a connected simply connected nilpotent Lie group and Γ is a discrete subgroup of $E(N)$. Then Γ acts freely (because $\Gamma \backslash N$ is a manifold) and properly discontinuously on N. M is a *nilmanifold* if in addition $\Gamma \subset N \subset E(N)$. See [19, § 6] for a discussion. Here we sharpen [19, Theorem 6.4] as follows.

Theorem 3.1. *Let M be a compact connected riemannian manifold with every mean curvature ≥ 0. Suppose that the underlying differentiable manifold of M is homotopy-equivalent to a compact generalized nilmanifold. Then M is flat, i.e., M is isometric to a compact euclidean space form. Further, the following conditions are equivalent*: (i) *M is a nilmanifold*; (ii) *$\pi_1(M)$ is nilpotent*; (iii) *M is a flat riemannian torus.*

Proof. Let N be a connected simply connected Lie group, and $\Gamma \subset E(N)$ a discrete subgroup such that there is a homotopy equivalence $f: M \to \Gamma \backslash N$. According to L. Auslander ([1]; or see [19, Proposition 6.2]) there is an exact sequence $1 \to \Sigma \to \Gamma \to \Psi \to 1$, where $\Sigma = \Gamma \cap N$ is a maximal nilpotent subgroup of Γ and Ψ is finite. Now f lifts to a homotopy equivalence $f: M' \to \Sigma \backslash N$ where M' is a finite riemannian covering manifold of M. From the proof of [19, Theorem 6.4], M' is diffeomorphic to a torus. Thus $b_1(M') = n$ where $n = \dim M' = \dim M$. Corollary 2.2 above says that M' is a flat riemannian torus. In particular M is flat.

Observe that Γ is nilpotent exactly when it coincides with $\Sigma = \Gamma \cap N$ because the latter is a maximal nilpotent subgroup. If M is a nilmanifold then $\Gamma \cong \pi_1(M)$ is nilpotent. If Γ is nilpotent then $M = M'$, a flat riemannian torus. If M is a flat riemannian torus then it is a nilmanifold $\boldsymbol{Z}^n \backslash \boldsymbol{R}^n$.

In particular, since euclidean space forms are generalized nilmanifolds, we have

Corollary 3.2. *Let M be a compact connected riemannian manifold with every mean curvature ≥ 0. If M is homotopy-equivalent to a compact euclidean space form, then M is flat.*

4. The Calabi construction for Ricci-flat manifolds

We now specialize to manifolds with every mean curvature zero, i.e., whose Ricci tensor $\boldsymbol{r} = 0$. We extend Calabi's result on the structure of compact euclidean space forms from flat manifolds to Ricci-flat manifolds. As a consequence, one of the Bieberbach theorems can also be generalized to the Ricci-flat case.

Our extension of the Calabi construction specifies the Ricci-flat n-manifolds in terms of the Ricci-flat manifolds of dimension $< n$ and the Ricci-flat n-manifolds with $b_1 = 0$. Similarly our extension of the Bieberbach theorem reduces the question of existence of nonflat Ricci-flat manifolds to the case of simply connected manifolds.

Using these results we give various sufficient topological conditions for Ricci-flat riemannian manifolds to be flat (Corollaries 4.3, 4.4; Theorem 4.6).

Part of our argument in generalizing the Calabi construction to the Ricci-flat case is the standard Selberg discontinuity technique [16, p. 149]. Yau uses that technique to obtain a weaker result [23, Theorem 3] under the weaker hypothesis that M have every mean curvature ≥ 0.

Theorem 4.1. *Let M^n be a compact connected Ricci-flat* ($\boldsymbol{r} = 0$) *riemannian n-manifold and $k = b_1(M^n)$. Then there is a finite normal riemannian covering*

$$p: T^k \times M^{n-k} \to M^n = \Psi \backslash (T^k \times M^{n-k})$$

where

(1) *T^k is a flat riemannian k-torus,*

(2) *$\Psi = \{(h(\varphi), \varphi) : \varphi \in \Phi\}$, where Φ is a finite group of isometries of M^{n-k} and h is an injective homomorphism of Φ into the translation group of T^k (so Ψ acts freely and properly discontinuously on $T^k \times M^{n-k}$),*

(3) *M^{n-k} is a compact connected Ricci-flat riemannian $(n-k)$-manifold which has no nonzero Φ-invariant parallel vector fields.*

Conversely, given T^k, M^{n-k}, and Ψ as above,

$$M^n = \Psi \backslash (T^k \times M^{n-k})$$

is a compact connected Ricci-flat riemannian n-manifold with first Betti number k, and M^n is determined up to affine equivalence by (M^{n-k}, Φ, k).

Proof. From Theorem 2.1, the identity component of the isometry group $\boldsymbol{I}(M)^0$ is the torus group T^k. Let $\pi: \tilde{M}^n \to M^n = \Gamma \backslash \tilde{M}^n$ be the universal riemannian covering. Γ is a discrete subgroup of the isometry group $\boldsymbol{I}(\tilde{M}^n)$. The torus group $\boldsymbol{I}(M^n)^0$ lifts to a real vector group $\boldsymbol{R}^k$ of ordinary translations along the euclidean factor in the de Rham decomposition of $\tilde{M}^n$. Thus $\tilde{M}^n = E^k \times \tilde{M}^{n-k}$, where E^k is a euclidean k-space and the $\boldsymbol{R}^k$-orbits are the $E^k \times \{m\}$, $m \in \tilde{M}^{n-k}$. This product splitting is stable under Γ because $\boldsymbol{R}^k$ centralizes

Γ. Since $I(M^n)^0 = R^k/R^k \cap \Gamma$ and is compact, $R^k \cap \Gamma$ is a lattice in R^k which is central in Γ. If $\gamma \in \Gamma$ then $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_1 \in I(E^n)$ and $\gamma_2 \in I(\tilde{M}^{n-k})$ because the product structure $E^k \times \tilde{M}^{n-k}$ is Γ-invariant. Define $\Gamma_i = \{\gamma_i : \gamma \in \Gamma\}$, so $\Gamma \subset \Gamma_1 \times \Gamma_2$. Since $R^k \cap \Gamma$ is a lattice in R^k and is central in Γ, γ_1 is an ordinary translation on E^k. Now Γ_1 is abelian, and its derived (commutator) group is $[\Gamma, \Gamma] = 1 \times [\Gamma_2, \Gamma_2]$. The quotient $\Gamma/[\Gamma, \Gamma] \cong H_1(M^n; Z)$ is the product of a finite abelian group with a finitely generated abelian group of Z-rank k. Since $Z^k \cong (R^k \cap \Gamma) \subset (\Gamma_1 \times 1)$, we have

$$\Delta = (R^k \cap \Gamma) \times [\Gamma_2, \Gamma_2]$$

is a normal subgroup of finite index in Γ. In particular $R^k \cap \Gamma$ has finite index in $\Gamma_1 \times 1$, and $[\Gamma_2, \Gamma_2]$ has finite index in Γ_2.

Define $A = \{\gamma \in \Gamma : \gamma_2 = 1\}$ and $B = \{\gamma \in \Gamma : \gamma_1 = 1\}$. Then $A = R^k \cap \Gamma$ because Γ_1 consists of translations of E^k. Evidently $(1 \times [\Gamma_2, \Gamma_2]) \subset B \subset (1 \times \Gamma_2)$. Now

$$\Sigma = A \times B = (R^k \cap \Gamma) \times B$$

is a normal subgroup of finite index in Γ. Define

$$T^k = (R^k \cap \Gamma)\backslash E^k\,, \quad M^{n-k} = B\backslash \tilde{M}^{n-k}\,, \quad \Psi = \Gamma/\Sigma\,.$$

Then T^k is a flat riemannian k-torus, M^{n-k} is a compact connected Ricci-flat riemanniann $(n-k)$-manifold, and the projection

$$p : T^k \times M^{n-k} \to \Psi\backslash(T^k \times M^{n-k}) = \Gamma\backslash\tilde{M}^n = M$$

is a finite normal riemannian covering.

Let $\psi \in \Psi$, say $\psi = \gamma AB$. Then ψ acts on T^k by a translation $\tau = \gamma_1 A$. If ψ is trivial in $T^k = A\backslash E^k$, then we replace γ by an element of γA and can assume $\gamma_1 = 1$. Consequently $\gamma \in B$, so $\psi = 1$. Similarly if ψ is trivial on M^{n-k}, then $\psi = 1$. Thus $\Psi = \{(h(\varphi), \varphi) : \varphi \in \Phi\}$, where Φ is a finite subgroup of $I(M^{n-k})$ and h is an injective homomorphism of Φ to the translation group of T^k.

If M^{n-k} has a nonzero Φ-invariant parallel vector field, then that field induces a parallel vector field Y on M^n. The lift of Y to $\tilde{M}^n$ must be tangent to E^k, contradicting the provenance of Y. Thus M^{n-k} has no nonzero Φ-invariant parallel vector fields.

Given T^k, M^{n-k} and Ψ as in the statement of the theorem, it is obvious that $M^n = \Psi\backslash(T^k \times M^{n-k})$ has the required properties.

Fix M^{n-k}, Φ and k as in the statement of the theorem. Let h_i be injective homomorphisms of Φ to the translation group of T^k. Define $\Psi_i = \{(h_i(\varphi), \varphi) : \varphi \in \Phi\}$. Since the h_i are injective and Φ is finite, there is an automorphism α of the translation group of T^k such that $h_2 = \alpha \cdot h_1$. Now $\alpha \times 1 : T^k \times M^{n-k}$

$\to T^k \times M^{n-k}$ induces an affine equivalence of $\Psi_1\backslash(T^k \times M^{n-k})$ onto $\Psi_2\backslash(T^k \times M^{n-k})$. q.e.d.

Roughly speaking, Theorem 4.1 says that modulo identifications from a finite group of isometries, it is possible to split off a flat k-dimensional torus $k = b_1(M)$ from a Ricci-flat riemannian manifold. This simplifies the topology and reduces the dimension of the spaces on which we study Ricci-flat metrics. To be precise, Theorem 4.1 reduces the affine classification of compact n-dimensional Ricci-flat manifolds to

(i) the classification in dimensions $< n$,

(ii) the classifications in dimension n with $b_1 = 0$, and

(iii) the classification of finite abelian groups Φ of isometries of compact Ricci-flat manifolds M^{n-k}, $0 \leq k < n$, such that M^{n-k} has no nonzero Φ-invariant parallel vector field.

Iterating Theorem 4.1 we obtain the following.

Corollary 4.2. *Let M^n be a compact connected Ricci-flat riemannian n-manifold. Then there is a series of finite normal riemannian coverings*

$$T^{k_r} \times M^{n-k_r} \to T^{k_{r-1}} \times M^{n-k_{r-1}} \to \cdots \to T^{k_1} \times M^{n-k_1} \to M$$

where $b_1(M^n) = k_1 < \cdots < k_r$, each M^{n-k_i} is a compact connected Ricci-flat riemannian $(n - k_i)$-manifold, each T^{k_i} is a flat riemannian k_i-torus, $b_1(M^{n-k_i}) = k_{i+1} - k_i$ for $1 \leq i < r$, and $b_1(M^{n-k_r}) = 0$.

As another corollary, we obtain the following result of Willmore [18] which generalizes the classical result that Ricci-flat riemannian manifolds of dimension ≤ 3 are flat.

Corollary 4.3. *Let M^n be a compact connected Ricci-flat riemannian n-manifold. If $b_1(M^n) \geq n - 3$, then M^n is flat.*

Proof. Applying Theorem 4.1, M^{n-k} is a Ricci-flat riemannian manifold of dimension ≤ 3, so from [12] it is flat. Hence M is flat. q.e.d.

Lichnerowicz [12, p. 219] and Yau [23, Corollary 1] prove Corollary 4.3 in the case $n = 4$.

Using the same technique as in Theorem 2.4, we can derive a weaker sufficient condition for Ricci-flat manifolds to be flat.

Corollary 4.4. *Let M be a compact connected n-dimensional manifold. Suppose that M has a finite topological covering $\pi : \tilde{M} \to M$ with $b_1(\tilde{M}) \geq n - 3$. Then every Ricci-flat riemannian structure on M is flat.*

Note that this weakening of the topological condition on $\tilde{M}$ compared to Theorem 2.4 is a consequence of our strengthening the geometrical condition on M.

According to Cheeger and Gromoll [8, Theorem 3], $\pi_1(M)$ has a finite normal subgroup P such that there is an exact sequence $1 \to Z^k \to \pi_1(M)/P \to$ (finite) $\to 1$. If we replace M by a finite covering we increase b_1 but evidently do not increase k. Doing that we may suppose $\pi_1(M)/P = Z^k$ with $k = b_1(M)$, and Theorem 4.1 then gives us $T^k \times M^{n-k} \to M$ with $\pi_1(M^{n-k})$ finite. If N is the universal riemannian covering of M^{n-k}, then we have

Theorem 4.5. *Let M be a compact connected Ricci-flat riemannian manifold. Then there is a flat riemannian torus T of dimension $\geq b_1(M)$, a compact simply connected Ricci-flat riemannian manifold N, and a finite riemannian covering $T \times N \to M$.*

This extends the Bieberbach theorem ([14]; see [20, Theorem 3.3.1]) which says that a compact euclidean space form admits a finite normal riemannian covering by a flat torus. This extension can also be extracted from the work of Cheeger-Gromoll [8], specifically from the proof of Theorem 6.

Remarks. 1. If $\pi_1(M)$ is infinite, then $\dim T \geq b_1(M) > 0$ so $\chi_M = 0$.

2. If M is flat, then Theorem 4.5 specializes to the Bieberbach theorem; N, being compact connected simply connected and flat, reduces to a point. If M is not flat, then $\dim N \geq 4$.

3. If every compact simply connected Ricci-flat manifold is flat, then the theorem shows that every compact Ricci-flat manifold is flat.

Using this extension of the Bieberbach Theorem, we can find some interesting sufficient topological conditions for Ricci-flat manifolds to be flat.

Theorem 4.6. *Let M be a compact connected Ricci-flat n-dimensional riemannian manifold. Then the following are equivalent:*

1. *M is flat.*
2. *For $k > 1$ the homotopy groups $\pi_k(M) = 0$.*
3. *The universal covering of M is acyclic.*

Proof. (1) $\Rightarrow$ (3) and (2). If M is flat, its universal covering $p: E^n \to M$ is a euclidean n-space which is contractible and hence acyclic. Also, $\pi_k(E^n) = 0$ for all $k \geq 1$, so $\pi_k(M) = 0$ for $k > 1$ by the isomorphism $p_*: \pi_k(E^n) \to \pi_k(M)$ for $k > 1$.

Not (1) $\Rightarrow$ Not (3) and Not (2). Suppose M is not flat. Then from Theorem 4.5, the universal covering of M is $p: E \times N \to M$, where E is a euclidean space and N is compact simply connected and of dimension $r \geq 4$. Then $H_r(N) = H_r(E \times N)$ is infinite cyclic, so the universal covering cannot be acyclic.

Now let s be the smallest positive integer such that $H_s(N) \neq 0$, $s \leq \dim N$. Since N is simply connected, $s \geq 2$, and by the Hurewicz isomorphism theorem $\pi_{s'}(N) = 0$ for $s' < s$ and $\pi_s(N) = H_s(N) \neq 0$. Thus $\pi_s(M) \cong \pi_s(E \times N) \cong \pi_s(N) \neq 0$. q.e.d.

Finally we comment that none of our results exclude the possibility that the Kummer surface [17], which is a compact simply connected 4-manifold with $b_2 = 22$ and $\chi = 24$, might carry a nonflat Ricci-flat riemannian metric.

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